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Determination of Control Pairing for Higher Order Multivariable Systems by the use of Multi-Ratios

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Abstract. The relative gain array (RGA) and the Niederlinski index (NI) are normally used in tandem to determine the control configuration for multivariable systems. For two-input, two-output systems, the RGA and the NI can be characterized by a single ratio, zeta (ζ) which is the ratio of the product of the off- diagonal terms to that of the diagonal terms of the steady state gain matrix. This paper extends the concept of the zeta ratio, developed for 2×2 systems to higher order systems, and subsequently uses it in place of the RGA to determine suitable control configurations for higher order multivariable systems. Several examples used to demonstrate the effectiveness of the proposed method for higher order systems show that it is computationally simpler and easier to understand and apply by control practitioners.

Index Terms— .Decentralized control, Multivariable .Ssystems, Niederlinski Index Relative Gain Array;; Zeta ratio

1 INTRODUCTION

Industrial processes normally require the control of two or more controlled output variables that relate to production rate, product quality, safety and environmental concerns. This in turn requires two or more manipulated input variables, thus giving rise to a multi-input, multi-output system (MIMO), or multivariable system. These multivariable systems are either controlled by a centralized controller or by a set of single-input single-output decentralized controllers. Decentralized control are more often used for process control applications because it is flexible, simple to design, implement and tune [1]. Decentralized control attempts to control the multivariable system by decomposing it into several single-input-single-output (SISO) control loops. In order to design a decentralized controller, it is necessary to appropriately pair the input and output variables so as to have minimal interactions from and to the other loops in the closed loop multivariable control system.

A lot of interaction measures have been developed to help determine the best variable pairing that would achieve minimal interaction [2], [3], [4], [5]. The Relative Gain Array, proposed by Bristol in 1966 still has the widest application in industry [6], [7], [8].

2 THEORETICAL PRINCIPLES

2.1 The Relative Gain Array (RGA)

Bristol's Relative Gain Array (RGA) [2] is a well established tool for analysis and design of MIMO control systems. Considering the closed loop multivariable system shown in Figure 1, where the process to be controlled has an equal number of input; u_i, i=1, 2,…,n and output variables; y_i, i=1, 2,…,n .
The process transfer function is given by:

$$y(s) = G(s)u(s) \tag{1}$$

where

$$\underline{y}(s) = (y_1(s), y_2(s), \ldots, y_n(s))^T,$$
$$\underline{u}(s) = (u_1(s), u_2(s), \ldots, u_n(s))^T$$

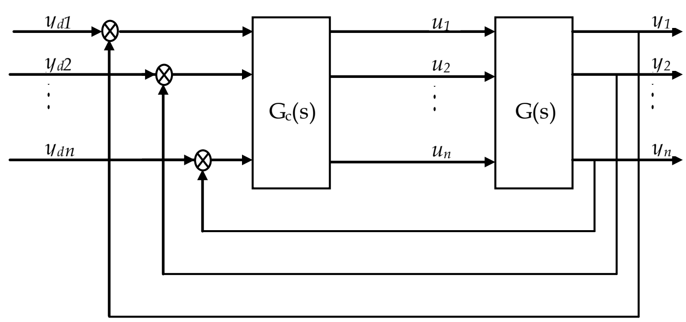

Figure 1. Block diagram of a closed loop multivariable control system

and G(s) the transfer function matrix of the process is given by

$$\underline{G}(s) = \begin{bmatrix} g_{11}(s) & \cdots & g_{1n}(s) \\ \vdots & \ddots & \vdots \\ g_{n1}(s) & \cdots & g_{nn}(s) \end{bmatrix}$$

and the decentralized controller matrix is given by

$$\underline{G}_c(s) = \begin{bmatrix} g_{c1(s)} & 0 & \cdots & 0 \\ 0 & g_{c2(s)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g_{cn(s)} \end{bmatrix}$$

where $\underline{G}_c(s)$ the diagonal matrix of SISO controllers designed based on the diagonal elements of the process transfer function, g_{ii} $(i = 1 \ldots n)$

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The RGA for a square system is defined as the matrix $\underline{\Lambda}$ such that the element λ_{ij} is determined as:

$$\lambda_{ij} = \frac{\left(\frac{dy_i}{du_j}\right)_{\text{all loops open}}}{\left(\frac{dy_i}{du_j}\right)_{\text{all loops open except loop } u_j}} \quad (2)$$

The RGA may be evaluated from the transfer-function matrix of a square multivariable system by doing a Hadamard or Schur product (element-by-element multiplication) of the transfer-function matrix $\underline{G}(s)$ and the transpose of the inverse of this matrix, $\underline{G}^{-T}$, where

$$\underline{G}^{-T}(s) = (\underline{G}^{T}(s))^{-1} = (\underline{G}^{-1}(s))^{T}.$$

The RGA is normally evaluated at steady-state, for which $\underline{G}(s)$ becomes $\underline{G}(0)$, that is $\underline{G}(s)\big|_{s=0}$ and

$$\underline{\Lambda} = \underline{G}(0) \otimes \underline{G}^{-T}(0) \text{ and } \lambda_{ij} = g(0)_{ij}\, g^{-T}(0)_{ij}$$

The closer λ_{ij} is to unity, the better it is to control the ith controlled output using the jth manipulated input. Therefore the best control configuration would be one in which the diagonal elements of the RGA are closest to unity, and the rest are closest to zero. Skogestad and Morari [10] and Chen *et. al*[10] and Smith and corripio [11] provide detailed discussions on the use of the RGA.

2.2 The Niederlinski Index [12]

The Niederlinski index determines the best control configuration for a system based on stability analysis. It is defined as:

$$NI = \frac{\left|\underline{G}(0)\right|}{\prod_{i=1}^{n} g_{ii}} \quad (3)$$

A negative value for NI, when all the control loops are closed, implies the system will be integrally unstable for all possible values of controller parameters.

In order to design a decentralized control system for a process, given the transfer function, the RGA is used to obtain a tentative loop pairing, then the NI is used to ascertain the stability of the closed loop system using the recommended RGA pairing, simulation runs are then used to verify if the recommended pairings are suitably stable.

2.3 Defining the zeta ratio for the 2x2 System [1]

Considering a two-input, two-output system whose steady state gain matrix is given by:

$$\underline{K} = \underline{G}(0) = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

and the RGA ,

$$\underline{\Lambda} = \frac{1}{\left|\underline{K}\right|}\begin{bmatrix} K_{11}K_{22} & -K_{12}K_{21} \\ -K_{12}K_{21} & K_{11}K_{22} \end{bmatrix}$$

while the NI is given by

$$NI = \frac{K_{11}K_{22} - K_{21}K_{12}}{K_{11}K_{22}}$$

Defining

$$\xi = \frac{K_{21}K_{12}}{K_{11}K_{22}}$$

and rewriting the RGA and NI in terms of ζ, the zeta ratio, gives

$$\underline{\Lambda} = \frac{1}{1-\xi}\begin{bmatrix} 1 & -\xi \\ -\xi & 1 \end{bmatrix} \quad \text{and} \quad NI = 1 - \xi$$

Both Λ and NI can therefore be said to be functions of ζ

Hence a 2×2 system can be fully characterized by the unique ratio ξ – the zeta ratio, nd the smaller the value of ξ, the more perfect is the diagonal pairing However, for higher order systems, there are many more ratios to be considered.

3 CONSIDERATIONS FOR HIGHER ORDER SYSTEMS:

3.1 The 3×3 System

For a 3×3 system with steady state gain matrix:

$$\underline{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$$NI = \frac{\left|\underline{K}\right|}{K_{11}K_{22}K_{33}}$$

This can be written fully as,

$$NI = \frac{K_{11}K_{22}K_{33} - K_{11}K_{32}K_{23} + K_{12}K_{23}K_{31} - K_{12}K_{21}K_{33} + K_{13}K_{21}K_{32} - K_{13}K_{22}K_{31}}{K_{11}K_{22}K_{33}} \quad (4)$$

Defining two other variables, E_{ij}, where ij are the various combinations possible for the 3×3 system (that is 12, 13, and 23) and D_{ijk}.

where, $$E_{ij} = \frac{K_{ij}K_{ji}}{K_{ii}K_{jj}}, \text{ and } D_{ijk} = \frac{K_{ij}K_{jk}K_{ki}}{K_{ii}K_{jj}K_{kk}}$$

Equation (4) can now be rewritten as

$$NI = 1 - E_{23} - E_{12} - E_{13} + D_{123} + D_{213} \quad (5)$$

and the Relative Gain Array, is given by

$$\underline{\Lambda} = \frac{1}{NI}\begin{bmatrix} 1-E_{23} & D_{123}-E_{12} & D_{213}-E_{13} \\ D_{213}-E_{12} & 1-E_{13} & D_{123}-E_{23} \\ D_{123}-E_{13} & D_{213}-E_{23} & 1-E_{12} \end{bmatrix}$$

$$= \frac{K_{ij} \times c_{ij}}{NI \times \prod_{1}^{n} K_{ii}}$$

where c_{ij} is the cofactor of k_{ij}

Hence a 3x3 system cannot be fully characterized by a unique ratio, as was done for the 2×2 system. It requires the five ratios outlined above. It can further be shown that the **number of ratios required for an n ×n system are *n*! – 1**, where *n* is the dimension of the square gain matrix.

But, as in the case of the 2×2 system, **the lower the ratio of the product of the non-diagonal elements in the transfer function matrix to the product of the diagonal ones**, **the less interactions there are in the system**. This was found to be applicable to all square systems regardless of the dimension.

This hypothesis was tested using a MATLAB program based on the algorithm below: (see program in Appendix):

1. Input the matrix dimension, *n*
2. Input the gain array elements
3. Generate all the possible single loop control configurations (*n*! combinations)
4. Evaluate the *NI* of each of the control configurations
5. For the configurations with *NI* > 0, determine the ξ value (ξ is the ratio of the product of the non-diagonal terms to the product of the diagonal ones in a gain matrix)
6. Sort the viable control configurations in order of increasing ξ value (The one with the least value is referred to as the zeta ratio and gives the suitable control configuration).
7. Evaluate the RGA matrix of the this configuration.

The effectiveness of the zeta ratio, ξ, for use in loop pairing in the design of decentralized multi loop controllers is investigated.

3.2 Case Examples:

This section tests the hypothesis and shows the effectiveness of using the zeta ratio.

Example 1

Using the 2×2 system in [13] in which McAvoy worked on the dynamic relative gain, DRGA - a modification to the RGA proposed by Tung and Edgar [4] and whose transfer function is given by

$$\underline{\mathbf{G}}(s) = \begin{bmatrix} \frac{5e^{-40s}}{100s+1} & \frac{e^{-4s}}{10s+1} \\ \frac{-5e^{-4s}}{10s+1} & \frac{5e^{-40s}}{100s+1} \end{bmatrix}.$$

The RGA obtained for this system $\underline{\Lambda} = \begin{bmatrix} 0.8333 & 0.1667 \\ 0.1667 & 0.8333 \end{bmatrix}$

with an NI of 1.2 recommends diagonal pairing, but as is expected due to the dynamic characteristics of the diagonal terms, with time constants and time delays are 10 times slower than the off-diagonal terms, poor closed loop performance was observed by McAvoy *et al* ([13]. However, as noted in [4] the off-diagonal pairing takes advantage of the fast dynamic characteristics to achieve better closed loop performances. However, the RGA is a steady-state analysis tool which does take the dynamics into consideration.

Using the multi-ratio concept, the program also recommends the off diagonal pairing of 1-2, 2-1 with an NI value of 6.0 and a zeta ratio, ξ = -0.2.

Example 2

Using the model given in [14]for which the transfer function matrix is:

$$\underline{\mathbf{G}}(s) = \begin{bmatrix} \frac{5}{4s+1} & \frac{2.5e^{-5s}}{(2s+1)(15s+1)} \\ \frac{-4e^{-6s}}{20s+1} & \frac{1}{3s+1} \end{bmatrix}$$

The RGA obtained for this system

$$\underline{\Lambda} = \begin{bmatrix} 0.3333 & 0.6667 \\ 0.6667 & 0.3333 \end{bmatrix}$$

with an NI of 3.0 recommends off-diagonal pairing. However, Xiong et. al. [15] in their work on effective RGA (ERGA), using this same model obtained better closed loop performance with diagonal pairing

The program also confirms the result of Xiong in that it recommends diagonal pairing with an NI value of 1.5 and a zeta ratio, ξ = -0.5.

Example 3

Also using another example used in [15]), a 3 × 3 process whose transfer function is given by

$$\underline{\mathbf{G}}(s) = \begin{bmatrix} \frac{-2e^{-s}}{10s+1} & \frac{1.5e^{-s}}{s+1} & \frac{e^{-s}}{s+1} \\ \frac{1.5e^{-s}}{s+1} & \frac{e^{-s}}{s+1} & \frac{-2e^{-s}}{10s+1} \\ \frac{e^{-s}}{s+1} & \frac{-2e^{-s}}{10s+1} & \frac{1.5e^{-s}}{s+1} \end{bmatrix}$$

and has a relative gain array of

$$\Lambda = \begin{bmatrix} -0.9302 & 1.1860 & 0.7442 \\ 1.1860 & 0.7442 & -0.9302 \\ 0.7442 & -0.9302 & 1.1860 \end{bmatrix}$$

From which there are two possible recommended pairings 1-2/2-1/3-3 and 1-3/2-2/3-1. Using Xiong's ERGA recommends 1-2/2-1/3-3 as the better of the two pairings but using the zeta ratio concept recommends the 1-3/2-2/3-1pairing, which has an NI = 1.5926 and a zeta ratio, ξ = -2.307.

Example 4

Considering a model with one of the elements changed to zero, such that the steady- state gain matrix is given by.

$$\underline{G}(0) = \begin{bmatrix} -4.19 & 0 & 1 \\ 1 & -25.96 & 6.19 \\ 1 & 1 & 1 \end{bmatrix}$$

The program failed because of the zero element value since in computing the zeta value, having a zero in the denominator gave an infinite value. To resolve this, a limiting value was substituted to prevent program failure.

The optimal configuration recommended by the program was the diagonal pairing, with RGA given by:

$$\underline{\Lambda} = \begin{bmatrix} 0.8332 & 0 & 0.1668 \\ 0.0062 & 0.8334 & 0.1604 \\ 0.1606 & 0.1666 & 0.6728 \end{bmatrix}$$

Example 5

Using the model in [16] whose steady-state gain matrix is given by

$$\underline{G}(0) = \begin{bmatrix} -0.64 & -0.21 & 1.82 \\ -0.6 & 1.19 & -0.34 \\ 0.55 & -1.12 & 1.14 \end{bmatrix}$$

and which gave unsatisfactory pairing using the RGA, in that the RGA recommended pairing returned a negative NI value of -2.8601. Using the zeta ratio concept, the program recommends the 1-2/2-1/3-3 pairing as optimal with zeta ratio of -2.021 and an NI = 4.8526.

Example 6:

The model reported in [17] for a complex distillation column with side stream stripper for separating ternary mixtures into 3 products is used for the 4 × 4 system. The output variables are the mole fractions of one of the components in each of the 3 phases and the change in temperature (ΔT) while the manipulated variables are the reflux ratio, the reboiler heat duty, the stripper heat duty and the stripper flow rate.

The steady state gain matrix obtained is:

$$\underline{G}(0) = \begin{bmatrix} 4.09 & -6.36 & -0.25 & -0.49 \\ -4.17 & 6.93 & -0.05 & 1.53 \\ 1.73 & 5.11 & 4.61 & -5.49 \\ -11.2 & 14 & 0.1 & 4.49 \end{bmatrix}$$

The program recommends a 1-2/2-4/3-1/4-3 pairing as optimal with a positive NI of 46.465 and a zeta value of -3.915 × 10^4.

4. ADVANTAGES AND DISADVANTAGES OF USING THE MULTI-ZETA RATIOs

The index is easier to compute than the RGA. It is also easier to analyze, since a single value , and not a matrix is being analyzed this is acceptable to control practitioners who seem averse to too much mathematics. A drawback would have been the programme failure when any of the elements in the gain matrix is zero. This difficulty is overcome by replacing the zero value with a limiting value, which for the MATLAB program, the function *eps* (which is equal to 2.204 x 10^{-16}) was used

5. CONCLUSION

An alternative scheme for determining the optimal control configuration for a multivariable system has been proposed. This method first determines all the control configurations that give a positive NI and then select the one with the lowest zeta ratio, ξ, as the optimal configuration. Comparison with examples based on the RGA or its modification show that the zeta ratio concept, which is much simpler to implement, gives satisfactory result.

APPENDIX– THE MATLAB PROGRAM

clear

%Input the dimension of the square gain matrix and the matrix elements.

%Enter matrix elements ONE AT A TIME ON A row-by-row and enter 'eps' for elements with zero %value

n = input('matrix dimension=')

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```
g=zeros(n,n)
for i=1:n
   for j=1:n
      g(i,j)=input('gain matrix element')
   end
end

%Compute the number of combinations, n-factorial (nf)
nf = 1
for j = 1:n
  nf = nf*j
end


%Generate all the n-factorial possible gain matrix arrangements in a 3-dimensional array
p = perms(1:n)
M = zeros(n,n,nf)
for k = 1:nf
  M(:,:,k) = g(:,p(k,:))
end

%Determine all the PI-control stable configurations by computing Niederlinski index of each arrangement
for t = 1:nf
  a = 1
  for k=1:n
     a = a*M(k,k,t)
  end
  ni(t) = det(M(:,:,t))/a
end

%Compute the zeta index for control configurations that have positive Niederlinski indices
for t = 1:nf
if ni(t)>0
a=1
b=1
 for i=1:n
           for j=1:n
             a=a*M(i,j,t)
           end
           b=b*M(i,i,t)
 end
zeta(t) =  a/(b^2)
else ni(t)<0
zeta(t)=NaN
end
end

% Compute the RGA for the best configuration

c=min(zeta)
r=zeta/c
w=find(r==1)
rga =M(:,:,w).*(inv(M(:,:,w)))'

% Display the suggested control gain matrix configuration
OPTCONF =M(:,:,w)
```

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